

UNIVERSALITY AT WEAK AND STRONG NON-HERMITICITY BEYOND THE ELLIPTIC GINIBRE ENSEMBLE

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ABSTRACT. We consider a non-Gaussian extension of the elliptic Ginibre ensemble of complex non-Hermitian random matrices by adding a trace squared term. This ensemble has correlated matrix entries and a non-determinantal joint density of eigenvalues. These features are absent in non-Hermitian Wigner type ensembles or normal random matrix ensembles with a potential. We provide a detailed analysis of the limit of weak non-Hermiticity introduced by Fyodorov, Khoruzhenko and Sommers, where the support of the limiting measure collapses to the real line. The limiting local correlations, however, extend to the complex plane, are determinantal, and interpolate between sine kernel and Ginibre statistics. In particular, this constitutes a first proof of universality for the interpolating kernel. Furthermore, in the limit of strong non-Hermiticity in the bulk of the spectrum, we recover Ginibre statistics from our ensemble as well, thus extending known universality results.

1. INTRODUCTION AND MAIN RESULTS

Despite its almost equally long history, the investigation of random matrices without symmetry constraints is less advanced than that of random matrices which are for instance symmetric, Hermitian or unitary. More general, one distinguishes in Random Matrix Theory (RMT) between random matrices with eigenvalues living on a one-dimensional set in the complex plane \mathbb{C} and matrices with “genuinely complex” eigenvalues. These parts of the theory are somewhat imprecisely called Hermitian and non-Hermitian RMT. This article focuses on the local statistics of eigenvalues in non-Hermitian RMT.

On a technical level, a major obstacle in the analysis of random matrices without symmetries is the lack of a spectral calculus, making it difficult to even define the matrix models that amount for a large part of Hermitian RMT. Indeed, densities on the matrix space $\mathbb{C}^{N \times N}$ of the form $Z^{-1} \exp[-N \text{Tr}(V(J))]$, $J \in \mathbb{C}^{N \times N}$ cannot be defined for a potential V by spectral calculus. Even for polynomials like $V(J) = J^4$, the function $\exp[-N \text{Tr}(V(J))]$ is not normalizable. For normal matrices J , i.e. $JJ^* = J^*J$ ($*$ denoting the adjoint), the spectral theorem is available and matrix models can be defined for rotationally invariant $V : \mathbb{C} \rightarrow \mathbb{R}$ of sufficient growth for large arguments.

The best understood model in non-Hermitian RMT is the Ginibre ensemble [20], which can be considered as standard Gaussian measure on $\mathbb{C}^{N \times N}$, that is the probability measure with density proportional to $\exp[-N \text{Tr}(JJ^*)]$. Equivalently, a random matrix from the Ginibre ensemble can be realized as $J := J_1 + iJ_2$, where $J_{1,2}$ are independent Hermitian matrices from the Gaussian Unitary Ensemble (GUE), i.e. the matrix distribution on the space of $N \times N$ Hermitian matrices with density proportional to $\exp[-N \text{Tr}(J_{1,2}^2)]$. The matrix J will (almost surely) not be unitarily diagonalizable, but a Schur decomposition can be used to

obtain the joint density of its eigenvalues z_1, \dots, z_N on \mathbb{C}^N as proportional to

$$\prod_{j \neq l} |z_j - z_l|^2 e^{-N \sum_{j=1}^N |z_j|^2}.$$

The eigenvalues form a two-dimensional Coulomb gas and their statistics can be efficiently analyzed using its determinantal, i.e. the fact that its correlation functions are given as determinants of matrices of fixed size built from a kernel K_N , which in turn can be studied using orthogonal polynomials in the complex plane. Doing so, it was found that in the large N limit, the eigenvalues of J fill the unit disc in the complex plane with uniform density (the so-called “circular law”). Also the limiting local correlations between close eigenvalues could be computed.

The elliptic Ginibre ensemble was introduced as an interpolation between Hermitian and non-Hermitian matrices by taking $J := \sqrt{1 + \tau} J_1 + i\sqrt{1 - \tau} J_2$, where $J_{1,2}$ are again independent GUE matrices, and $\tau \in [0, 1)$ controls the degree of Hermiticity. The Ginibre ensemble is recovered choosing $\tau = 0$ and the GUE is obtained by formally taking the limit $\tau \rightarrow 1$. The elliptic ensemble has the density

$$P_{N,\text{ell}}(J) := \frac{1}{Z_{N,\text{ell}}} \exp \left[-\frac{N}{1 - \tau^2} \text{Tr} \left(J J^* - \frac{\tau}{2} (J^2 + J^{*2}) \right) \right] \quad (1)$$

on $\mathbb{C}^{N \times N}$, where $Z_{N,\text{ell}}$ is the normalizing constant. The reader will note the resemblance with the bivariate normal distribution. In this interpretation, τ may be understood as correlation coefficient between $\Re J_{j,l}$ and $\Re J_{l,j}$ or $\Im J_{j,l}$ and $\Im J_{l,j}$, $j \neq l$. We will focus on τ between 0 and 1, the case $\tau \in (-1, 0)$ can be treated analogously. The eigenvalue distribution corresponding to (1) is again determinantal, known in closed form (cf. (26)) and has been analyzed in great detail, cf. [3, Chapter 18] and references therein. As $N \rightarrow \infty$, its eigenvalues spread uniformly in the set

$$E := \left\{ Z : \left(\frac{\Re Z}{1 + \tau} \right)^2 + \left(\frac{\Im Z}{1 - \tau} \right)^2 \leq 1 \right\},$$

a fact that is termed “elliptic law”. Its limiting local correlations, however, coincide for $\tau \in [0, 1)$ fixed with those of the Ginibre ensemble.

Further ensembles in the literature are given by densities proportional to

$$\exp[-\sigma N \text{Tr}(J J^*) + \Re \text{Tr}(\Phi(J))], \quad (2)$$

where $\sigma > 0$ and Φ is a potential. The ensembles (2) are determinantal, but over $\mathbb{C}^{N \times N}$ the density is normalizable only for very specific potentials Φ . Basically, these are either $\Phi(J) = J^2$, corresponding to the elliptic ensemble, or logarithmic with special coefficient, see the discussion in [3, Chapter 39]. This problem can be circumvented by considering the density on a sufficiently large compact set instead of $\mathbb{C}^{N \times N}$ ([16], see also [11]).

The normal matrix models mentioned above also have determinantal eigenvalue distributions and belong to the same (bulk) universality class as the Ginibre ensemble, that is, their local correlations (in the bulk) coincide for $N \rightarrow \infty$ with those of the Ginibre ensemble [10, 7]. The same limiting local correlations have also been found for the non-Hermitian analogs of Wigner matrices [25], where (limiting) Ginibre correlations were shown for random matrices

with independent entries (and with independent real and imaginary parts) that have exponentially decaying distributions and fulfill certain moment conditions. In all these situations, one speaks nowadays of strong non-Hermiticity, meaning that the anti-Hermitian part of the random matrix is of the same order in N as the Hermitian part. This results in a limiting global distribution (the weak limit of the empirical spectral distribution $N^{-1} \sum_{j=1}^N \delta_{z_j}$) that is supported on a two-dimensional subset of \mathbb{C} . It is believed that the Ginibre correlations are universal for large classes of non-Hermitian random matrices with complex entries, very similar to the universality of sine kernel correlations for Hermitian random matrices.

Strong non-Hermiticity has to be contrasted with the so-called limit of weak non-Hermiticity, a situation where the limiting global distribution of the non-Hermitian random matrix is supported on the real line but the local correlations still extend to the complex plane. For the elliptic ensemble, this happens if the parameter τ is chosen as $\tau = 1 - \kappa/N$ for some $\kappa > 0$ not depending on N . The limit of weak non-Hermiticity was first discussed perturbatively in [18]. The limiting point process is determinantal and its kernel was derived in terms of Hermite polynomials in the complex plane in [17], see [19] for details. The limit of weak non-Hermiticity allows to describe the transition between sine kernel and Ginibre correlations. More precisely, it is a one-parameter deformation of the sine-kernel. This makes its universality highly suggestive, but so far only heuristic arguments in favour of this conjecture exist. For matrix ensembles with independent entries, these arguments can be found in [19], whereas the ensembles (2) were treated in [2].

The question of universality for matrix models that are not of the form (2) or correspond to normal matrices at strong non-Hermiticity and, in particular, at weak non-Hermiticity is thus open. The latter limit is of special importance in physics applications as it allows to relate to an effective field theory picture, which makes the random matrix approximation to the physics problem transparent. Our goal is thus, first, to make the derivation of the weak non-Hermiticity limit of [19] in the bulk rigorous, and second, to prove the universality for a new ensemble, both at weak and strong non-Hermiticity. Non-Hermitian random matrix theory enjoys many important applications in physics, including open quantum systems or quantum field theory with chemical potential [3]. Therefore it is important to understand the question of universality not only from a conceptual point of view.

Let us define the model which we will study in this paper. It is given by the density on $\mathbb{C}^{N \times N}$

$$P_{N, \text{Tr}^2}(J) := \frac{1}{Z_{N, \text{Tr}^2}} \exp \left[-\frac{N}{1-\tau^2} \text{Tr} \left(JJ^* - \frac{\tau}{2}(J^2 + J^{*2}) \right) - \gamma (\text{Tr} JJ^* - NK_p)^2 \right], \quad (3)$$

where $\gamma \geq 0$, $K_p \in \mathbb{R}$ are fixed, $\tau \in [0, 1)$ and Z_{N, Tr^2} is the normalization constant. This model resembles the elliptic ensemble (1) but has an additional penalization of deviations of $\text{Tr} JJ^*$ from the value NK_p . If the strength of the penalization γ goes to infinity, we formally arrive at a fixed trace ensemble, that is a probability measure on the set $\{J \in \mathbb{C}^{N \times N} : \text{Tr} JJ^* = NK_p\}$. P_{N, Tr^2} rather puts a “soft constraint” on $\text{Tr} JJ^*$. In contrary to the elliptic ensemble, obtained at $\gamma = 0$, P_{N, Tr^2} is for $\gamma > 0$ non-Gaussian and non-determinantal (cf. Remark 6 below).

For a probability density P on \mathbb{C}^N define the k -th correlation function

$$\rho^k(z_1, \dots, z_k) := \frac{N!}{(N-k)!} \int_{\mathbb{C}^{N-k}} P(z) dz_{k+1} \dots dz_N. \quad (4)$$

The correlation functions are multiples of the marginal densities. Let ρ_{N,Tr^2}^k denote the k -th correlation function of the eigenvalue density corresponding to P_{N,Tr^2} . We are now ready to state our main results. We start with the strongly non-Hermitian situation.

Theorem 1 (Limit of strong non-Hermiticity). *Let $\tau \in [0, 1)$ be fixed. Then there are constants $c_1, c_2, C > 0$, depending on K_p, γ and τ such that with $E := \{Z : c_1(\Re Z)^2 + c_2(\Im Z)^2 \leq 1\}$ the following holds:*

a) *For any $Z \in \mathbb{C}$, $Z \notin \partial E$, we have*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \rho_{N,\text{Tr}^2}^1(Z) = \mathbb{1}_{E^\circ}(Z) \cdot \frac{C}{\pi}.$$

b) *For $k = 1, 2, \dots$, $Z \in E^\circ$, $z_1, \dots, z_k \in \mathbb{C}$, as $N \rightarrow \infty$*

$$\frac{1}{(CN)^k} \rho_{N,\text{Tr}^2}^k \left(Z + \frac{z_1}{\sqrt{CN}}, \dots, Z + \frac{z_k}{\sqrt{CN}} \right) = \det (K_{\text{strong}}(z_j, z_l))_{j,l \leq k} + \mathcal{O} \left(\frac{1}{\sqrt{N}} \right),$$

where

$$K_{\text{strong}}(z_j, z_l) := \frac{1}{\pi} \exp \left[-\frac{|z_j|^2 + |z_l|^2}{2} + z_j \overline{z_l} \right]. \quad (5)$$

The \mathcal{O} term is uniform for Z from any compact subset of E° and z_1, \dots, z_k from compacts of \mathbb{C} .

Remark 2.

- a) The previous theorem shows in particular that the trace-squared ensemble (3) belongs to the same strongly non-Hermitian bulk universality class as the Ginibre ensemble.
- b) The constant C and the elliptic set E are given in Proposition 9. The set E differs from the limiting support of the elliptic ensemble, except if $K_p = 1$ or $\gamma = 0$, in which case $C = (1 - \tau^2)^{-1}$.
- c) In [25] for the Ginibre ensemble and [24] for the elliptic ensemble, the convergence is shown to be exponentially fast. In view of these results, it is likely that the bounds on the rate of convergence in Theorem 1 can be improved. As this is not one of the main purposes of this work, we will not pursue this here.
- d) Of course, the model (3) can also be considered on the set of $N \times N$ normal matrices. All our results extend to this situation as well. Note that when considered as a normal matrix model, the joint density of the eigenvalues is proportional to

$$\prod_{j < l} |z_j - z_l|^2 \exp \left[-\frac{N}{1 - \tau^2} \left(\sum_{j=1}^N |z_j|^2 - \frac{\tau}{2} \left(\sum_{j=1}^N z_j^2 + \overline{z_j}^2 \right) \right) - \gamma \left(\sum_{j=1}^N |z_j|^2 - NK_p \right)^2 \right].$$

When considered on $\mathbb{C}^{N \times N}$, the model does not have the same eigenvalue distribution.

We continue with the weakly non-Hermitian situation. Note the slightly differing values of τ in parts a) and b) of the following theorem.

Theorem 3 (Limit of weak non-Hermiticity). *There is a constant $C > 0$, depending on K_p and γ , such that the following holds:*

a) Let $\tau = \tau_N = 1 - \frac{\kappa}{N}$ with $\kappa > 0$ fixed. Then for any $Z \in \mathbb{C} \setminus \mathbb{R}$

$$\lim_{N \rightarrow \infty} \frac{1}{N} \rho_{N, \text{Tr}^2}^1(Z) = 0, \quad (6)$$

and for any $X \in \mathbb{R}$

$$\lim_{N \rightarrow \infty} \int_{\mathbb{R}} \frac{1}{N} \rho_{N, \text{Tr}^2}^1(X + iY) dY = \frac{C}{2\pi} \sqrt{\frac{4}{C} - X^2} \mathbb{1}_{[-\frac{2}{\sqrt{C}}, \frac{2}{\sqrt{C}}]}(X). \quad (7)$$

b) Set $\nu(X) := \frac{C}{2\pi} \sqrt{\frac{4}{C} - X^2}$ and $\tau = \tau_N := 1 - \frac{\alpha^2}{2N\nu(X)^2}$, $\alpha > 0$. Then for $k = 1, 2, \dots$, as $N \rightarrow \infty$

$$\begin{aligned} & \frac{1}{(N\nu(X))^{2k}} \rho_{N, \text{Tr}^2}^k \left(X + \frac{z_1}{N\nu(X)}, \dots, X + \frac{z_k}{N\nu(X)} \right) \\ &= \det(K_{\text{weak}}(z_j, z_l))_{j,l=1,\dots,k} + \mathcal{O}\left(\frac{\log N}{N}\right), \end{aligned}$$

where we denote $z_j := x_j + iy_j$ and define

$$K_{\text{weak}}(z_1, z_2) := \frac{\sqrt{2}}{\sqrt{\pi}\alpha} \exp\left[-\frac{y_1^2 + y_2^2}{\alpha^2}\right] \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\left[-\frac{\alpha^2 u^2}{2} + iu(z_1 - \bar{z}_2)\right] du.$$

The \mathcal{O} term is uniform for $X \in (-\frac{2}{\sqrt{C}} + \delta, \frac{2}{\sqrt{C}} - \delta)$ for any $\delta > 0$ fixed and any $z_j, j = 1, \dots, k$ chosen from an arbitrary compact subset of \mathbb{C} .

Remark 4.

- a) Part a) of the previous theorem shows that the support of the limiting measure collapses to the real axis whenever τ is in any $1/N$ neighborhood of 1. Moreover, the limiting marginal density of the real part X is the semicircle density $\nu(X)$, which is used in part b) to rescale not only the z_j 's, but also τ , in order to make the limiting kernel independent of X .
- b) K_{weak} is comparable to the kernel given in [19], where $x_1 = -x_2$ was chosen. We have simply rescaled α to make the kernel independent of X . The non-dependence of K_{weak} on γ and K_p constitutes a first universality result for the limit of weak non-Hermiticity.
- c) It is not hard to show that in the limit $\alpha \rightarrow 0$ we obtain

$$\begin{aligned} & \lim_{\alpha \rightarrow 0} \int_{\mathbb{C}^k} f(z_1, \dots, z_k) \det(K_{\text{weak}}(z_j, z_l))_{j,l=1,\dots,k} dz_1 \dots dz_k \\ &= \int_{\mathbb{R}^k} f(x_1, \dots, x_k) \det\left(\frac{\sin(\pi(x_j - x_l))}{\pi(x_j - x_l)}\right)_{j,l=1,\dots,k} dx_1 \dots dx_k \end{aligned}$$

for any bounded and continuous function $f : \mathbb{C}^k \rightarrow \mathbb{R}$ of bounded support.

For the limit $\alpha \rightarrow \infty$, we need to rescale the variables z_j with α in order to account for the difference in the local scales (in N) for the limits of weak and strong non-Hermiticity. Here it is straightforward to get

$$\lim_{\alpha \rightarrow \infty} \det(\alpha^2 K_{\text{weak}}(\alpha z_j, \alpha z_l))_{j,l=1,\dots,k} = \det(K_{\text{strong}}(z_j, z_l))_{j,l=1,\dots,k}.$$

- d) In [23], the first correlation function of the elliptic ensemble $\rho_{N,\text{ell}}^1(X + \frac{iy}{N})$ was studied in the limit of weak non-Hermiticity. Here, the variable on the real axis is in the global scaling regime whereas the imaginary variable is in the local scaling regime. With this particular choice of variables, one sees a transition from the semicircle law to the circular law.
- e) The constant C in Theorem 3 is given by

$$C = \frac{1}{2} - 2\gamma K_p + \frac{1}{2}\sqrt{16\gamma^2 K_p^2 - 8\gamma K_p + 16\gamma + 1}.$$

In particular, if $\gamma = 0$ or $K_p = 1$, then $C = 1$.

Let us finish this section with some concluding remarks. It was found by Bender that a similar weakly non-Hermitian scaling limit can be defined at the edge of the spectrum [9]. Here, for the largest real eigenvalue an interpolation between the Tracy-Widom and Gumbel distribution was found, where the latter corresponds to the Fredholm determinant of the complementary error function kernel of the Ginibre ensemble. The same interpolating kernel was found in a chiral variant of the Ginibre ensemble [4] and it was shown to be a one-parameter deformation of the Airy-kernel. We refer to [6] for an entire list of one-parameter deformed kernels at the edge, in the bulk and at the origin corresponding to the limits in Ginibre ensembles and its chiral partners with real, complex or quaternion matrix elements, see also references therein.

We would also like to remark that there is a considerable interest in 2D-Coulomb gases with external fields at various temperatures. These so-called (two-dimensional) β -ensembles directly describe a system of N particles on \mathbb{C} , only at certain temperatures ($\beta = 2, 4$) these particle systems correspond to eigenvalue distributions of normal matrix models. A full account of the literature on this subject seems to be outside of the scope of the present work, instead we point the reader to a few contributions and the references therein. Fluctuations of such β -ensembles around their macroscopic limits have been considered up to the finest possible scale in [21, 22]. Fluctuations of linear statistics have been studied in [8], whereas edge universality has been shown in [12].

Let us add a further comment on the non-Gaussian deformation we consider. In [5] they were used for Hermitian random matrices to represent fixed trace ensembles in a limiting procedure. Indeed trace squared terms favor an average value for the second moment of the empirical spectral measure and thus leads to a “soft constraint”. We defer the study of a fixed trace version of the elliptic Ginibre ensemble as an interesting problem to future work. To date only the finite N density of the expected spectral distribution of the fixed trace complex Ginibre ensemble is known [14]. As an additional motivation to study our model, there are also good field theoretic reasons to add a trace squared term as has been described in [13] in a Hermitian random matrix model. In analytic continuations to formal matrix models these additions can lead to phase transitions [13], which we will not study here.

Our article is organized as follows. In Section 2 we recenter our ensemble and linearize the trace squared term at the expense of an additional integral. The asymptotics of the linearized ensemble are then derived in Section 3, which we will use to prove our main theorems in Section 4.

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2. RECENTERING AND LINEARIZATION

We will start our analysis of the ensemble (3) by a recentering of the density which takes into account a part of $\gamma(\text{Tr}JJ^*)^2$ that has an influence on the global distribution of the eigenvalues. It is not hard to see that $\lim_{N \rightarrow \infty} \mathbb{E}_{N,\text{ell}} \text{Tr}(JJ^*)/N = 1$, where $\mathbb{E}_{N,\text{ell}}$ denotes expectation w.r.t. the elliptic ensemble (1). Hence for $K_p = 1$ the trace constraint does not change much and the global asymptotics (such as the support of the limiting ellipse) of the ensembles $P_{N,\tau}^{\gamma,1}$ and (1) should coincide. Thus in this case no recentering should be necessary as the penalization just reinforces the convergence of $N^{-1}\text{Tr}JJ^*$ to K_p . In contrast, choosing K_p different from 1 enforces $N^{-1}\text{Tr}JJ^*$ to have a different limit, which should lie between K_p and 1. In this case it will be convenient to renormalize the ensemble. To this end, let us for $a > b \geq 0$ consider the family of densities $P_{a,b}$ with

$$P_{a,b}(J) := \frac{1}{Z_{a,b}} \exp \left[-a \text{Tr}JJ^* + \frac{b}{2} \text{Tr} (J^2 + J^{*2}) \right] \quad (8)$$

and normalization constant $Z_{a,b}$. The motivation for introducing (8) comes from the following manipulation of P_{N,Tr^2} . Let $K > -(2\gamma(1+\tau))^{-1}$ and rewrite

$$\begin{aligned} P_{N,\text{Tr}^2}(J) &= \frac{\exp \left[-N \left(\frac{1}{1-\tau^2} + 2\gamma K \right) \text{Tr}JJ^* + \frac{\tau N}{2(1-\tau^2)} \text{Tr} (J^2 + J^{*2}) - \gamma(\text{Tr}JJ^* - N(K_p + K))^2 \right]}{Z_{N,\text{Tr}^2} \exp [-\gamma N^2(K^2 + 2KK_p)]} \\ &= \frac{Z_{a,b}}{Z_{N,\text{Tr}^2} \exp [-\gamma N^2(K^2 + 2KK_p)]} P_{a,b}(J) \exp [-\gamma(\text{Tr}JJ^* - N(K_p + K))^2] \end{aligned} \quad (9)$$

with

$$a := N \left(\frac{1}{1-\tau^2} + 2\gamma K \right), \quad b := \frac{\tau N}{1-\tau^2}.$$

For the rest of the paper, we stick to this choice of a and b . Note that the condition $K > -(2\gamma(1+\tau))^{-1}$ ensures that $a > b$, and thus the normalizability of $P_{a,b}$. Furthermore, comparing (9) and (8) shows that

$$\frac{Z_{N,\text{Tr}^2} \exp [-\gamma N^2(K^2 + 2KK_p)]}{Z_{a,b}} = \mathbb{E}_{a,b} \exp \left[-\gamma(\text{Tr}\tilde{J}\tilde{J}^* - N(K_p + K))^2 \right], \quad (10)$$

where $\mathbb{E}_{a,b}$ denotes expectation w.r.t. $P_{a,b}$ and we use the following convention throughout the paper: In equations, the matrix \tilde{J} is an integration variable, whereas J denotes an arbitrary, but fixed matrix. However, as there is no ambiguity, we will also let J denote the random matrix associated to a specified ensemble.

The following lemma gives the optimal choice of the so far arbitrary K in (9). In short, we determine K such that the statistic $\text{Tr}JJ^* - N(K_p + K)$ is concentrated under $P_{a,b}$.

Lemma 5. For each $\tau \in [0, 1)$, $\gamma \geq 0$ and $K_p \in \mathbb{R}$, there is a unique $K = K(\tau) = K(\tau, \gamma, K_p)$ such that for some constants C_1, C_2 independent of τ and N , we have for all N

$$0 < C_1 \leq \mathbb{E}_{a,b} \exp \left[-\gamma (\text{Tr} \tilde{J} \tilde{J}^* - N(K_p + K))^2 \right] \leq C_2 < \infty. \quad (11)$$

The same bound with the same constants C_1, C_2 holds true if $\lim_{N \rightarrow \infty} \tau_N = 1$. K is the unique positive zero of the cubic equation (20). The limit $\bar{K} := \lim_{\tau \rightarrow 1} K(\tau)$ exists,

$$\bar{K} > -\frac{1}{4\gamma}$$

and

$$|K(\tau) - \bar{K}| = \mathcal{O}(|\tau - 1|) \text{ as } \tau \rightarrow 1. \quad (12)$$

Proof. It is easy to check that under $P_{a,b}$, $\{\Re J_{j,k}, \Im J_{j,k}\}_{j,k}$ are jointly Gaussian random variables with mean 0 and covariance structure as follows. Diagonal entries $\Re J_{j,j}, \Im J_{j,j}$ have the variances

$$\sigma_{D,\Re}^2 := \frac{1 + \tau}{2N(1 + 2\gamma K(1 + \tau))}, \quad \sigma_{D,\Im}^2 := \frac{1 - \tau}{2N(1 + 2\gamma K(1 - \tau))}, \quad (13)$$

respectively, off-diagonal entries $\Re J_{j,k}, \Im J_{j,k}, j \neq k$ the variance

$$\sigma_O^2 := \frac{1 + 2\gamma K(1 - \tau^2)}{2N(1 + 4\gamma K + 4\gamma^2 K^2(1 - \tau^2))} \quad (14)$$

and the covariances are 0 except for ($j \neq k$)

$$\rho := \text{Cov}(\Re J_{j,k}, \Re J_{k,j}) = \frac{\tau}{2N(1 + 4\gamma K + 4\gamma^2 K^2(1 - \tau^2))} \quad (15)$$

and

$$\text{Cov}(\Im J_{j,k}, \Im J_{k,j}) = -\frac{\tau}{2N(1 + 4\gamma K + 4\gamma^2 K^2(1 - \tau^2))} = -\rho. \quad (16)$$

The matrix J can be associated with a vector $\underline{J} \in \mathbb{R}^{2N^2}$ that has a multivariate normal distribution with mean 0 and covariance matrix Σ that is of block diagonal form. Each block consists either of $\sigma_{D,\Re}^2$, $\sigma_{D,\Im}^2$, or 2×2 matrices with σ_O^2 on the diagonal and $\pm \rho$ off the diagonal. \underline{J} has the same distribution as $\underline{U}\underline{Y}$ where \underline{U} is a $2N^2 \times 2N^2$ unitary matrix and \underline{Y} is Gaussian with independent components with mean 0. The variances are $\sigma_{D,\Re}^2$ or $\sigma_{D,\Im}^2$ for N components each, and λ_+^2 or λ_-^2 for $N(N-1)$ components each, where

$$\lambda_{\pm}^2 := \sigma_O^2 \pm \rho \quad (17)$$

are the eigenvalues of those blocks of Σ that contain off-diagonal entries. As $\text{Tr}(JJ^*) = \|\underline{J}\|_2^2 \stackrel{d}{=} \|\underline{Y}\|_2^2$, where $\stackrel{d}{=}$ means equality of distributions, we find that

$$\text{Tr}(JJ^*) \stackrel{d}{=} \lambda_+^2 Z_1 + \lambda_-^2 Z_2 + \sigma_{D,\Re}^2 Z_3 + \sigma_{D,\Im}^2 Z_4, \quad (18)$$

where Z_1, Z_2, Z_3, Z_4 are independent χ^2 distributed random variables with $N(N-1), N(N-1), N$ and N degrees of freedom, respectively.

Let us now choose $K = K(\tau, \gamma, K_p)$ such that

$$K_p + K = 2N\sigma_O^2,$$

in other words

$$K_p + K = \frac{1 + 2\gamma K(1 - \tau^2)}{1 + 4\gamma K + 4\gamma^2 K^2(1 - \tau^2)}. \quad (19)$$

It is not hard to see that the r.h.s. has poles at $-(2\gamma(1 \pm \tau))^{-1}$ and is strictly decreasing in K for $K > -(2\gamma(1 + \tau))^{-1}$. As the l.h.s. of (19) is a strictly increasing continuous function on \mathbb{R} , there is precisely one $K > -(2\gamma(1 + \tau))^{-1}$ satisfying (19). It is the rightmost (real) solution of the cubic equation

$$4\gamma^2(1 - \tau^2)K^3 + 4\gamma(1 + \gamma(1 - \tau^2)K_p)K^2 + (1 + 4\gamma K_p - 2\gamma(1 - \tau^2))K + K_p - 1 = 0. \quad (20)$$

It can be computed explicitly but its exact form is not important here.

As $\tau \rightarrow 1$, (19) becomes

$$K_p + K = \frac{1}{1 + 4\gamma K}, \quad (21)$$

which has, analogously to (19), only one solution $\bar{K} := \lim_{\tau \rightarrow 1} K(\tau)$ that satisfies $\bar{K} > -\frac{1}{4\gamma}$. It is given by

$$\bar{K} = -\frac{K_p}{2} - \frac{1}{8\gamma} + \frac{1}{8\gamma} \sqrt{16\gamma^2 K_p^2 - 8\gamma K_p + 16\gamma + 1}.$$

Furthermore, by applying the implicit function theorem to the function

$$F(\tau, K) := 4\gamma^2(1 - \tau^2)K^3 + 4\gamma(1 + \gamma(1 - \tau^2)K_p)K^2 + (1 + 4\gamma K_p - 2\gamma(1 - \tau^2))K + K_p - 1,$$

we find that the derivative $K'(\tau)$ is bounded in a neighborhood of $\tau = 1$ and thus (12) follows.

Let us turn to proving (11). We have by (18)

$$\text{Tr}(JJ^*) - N(K_p + K) \stackrel{d}{=} \bar{\lambda}_+^2 \sqrt{2} \frac{Z_1 - N(N-1)}{\sqrt{2N^2}} + \bar{\lambda}_-^2 \sqrt{2} \frac{Z_1 - N(N-1)}{\sqrt{2N^2}} + \sigma_{D,\Re}^2 Z_3 + \sigma_{D,\Im}^2 Z_4, \quad (22)$$

where $\bar{\lambda}_\pm^2 := 2N\lambda_\pm^2$ do not explicitly depend on N (they may still depend on N via $\tau = \tau_N$). Now $(Z_j - N(N-1))/\sqrt{2N^2}$, $j = 1, 2$ converge weakly towards standard normals as $N \rightarrow \infty$ and $\sigma_{D,\Re}^2 Z_3, \sigma_{D,\Im}^2 Z_4$ converge weakly to constants. Since Z_i , $i = 1, 2, 3, 4$ are independent and $\bar{\lambda}_\pm$ converge in the case $\tau_N \rightarrow 1$, we have by Slutsky's theorem weak convergence of (22) to a Gaussian distribution and hence

$$\lim_{N \rightarrow \infty} \mathbb{E}_{a,b} \exp \left[-\gamma (\text{Tr} \tilde{J} \tilde{J}^* - N(K_p + K))^2 \right] = \exp \left[-\frac{C\gamma}{1 + c\gamma} \right] (1 + c\gamma)^{-1/2}$$

for some $C, c > 0$, i.e. convergence to the moment-generating function of a non-central chi-squared distribution. This proves (11). \square

From now on, let K always denote the quantity from Lemma 5. The linearization uses the simple identity

$$\exp [-\gamma X^2] = \frac{1}{\sqrt{4\pi}} \int_{\mathbb{R}} \exp [i\sqrt{\gamma} X t] \exp \left[-\frac{t^2}{4} \right] dt, \quad (23)$$

valid for any real X . In the physics literature, this is known as the Hubbard-Stratonovich transform (in its simplest form). With (23), we may rewrite P_{N, Tr^2} as

$$P_{N, \text{Tr}^2}(J) = \frac{1}{\sqrt{4\pi}} \int_{\mathbb{R}} \frac{\exp[-i\sqrt{\gamma}N(K_p + K)t]}{Z_{N, \text{Tr}^2} \exp[-\gamma N^2(K^2 + 2KK_p)]} \\ \times \exp\left[(-a + i\sqrt{\gamma}t) \text{Tr} J J^* + \frac{b}{2} \text{Tr}(J^2 + J^{*2})\right] \exp[-t^2/4] dt.$$

Setting

$$a(t) := a + i\sqrt{\gamma}t,$$

we have (extending the definition (8) to the complex $a(t)$)

$$\frac{Z_{a(t),b}}{Z_{a,b}} = \mathbb{E}_{a,b} \exp\left[i\sqrt{\gamma}t \text{Tr} \tilde{J} \tilde{J}^*\right],$$

and thus by (18), $Z_{a(t),b}/Z_{a,b}$ is the product of characteristic functions of χ^2 distributed random variables. Thus the function

$$t \mapsto \mathbb{E}_{a,b} \exp\left[i\sqrt{\gamma}t \text{Tr} \tilde{J} \tilde{J}^*\right] \quad (24)$$

has no zeros on the real line and the “linearized ensemble”

$$P_{a(t),b}(J) = \frac{1}{Z_{a(t),b}} \exp\left[\left(-N\left(\frac{1}{1-\tau^2} + 2\gamma K\right) + i\sqrt{\gamma}t\right) \text{Tr} J J^* + \frac{\tau N}{2(1-\tau^2)} \text{Tr}(J^2 + J^{*2})\right]$$

is well-defined. The term “ensemble” here is only a convenient naming, in general $P_{a(t),b}$ is complex-valued. Summarizing, we arrive at

$$P_{N, \text{Tr}^2}(J) = \frac{1}{\sqrt{4\pi}} \int_{\mathbb{R}} \frac{Z_{a(t),b} \exp[-i\sqrt{\gamma}N(K_p + K)t]}{Z_{N, \text{Tr}^2} \exp[-\gamma N^2(K^2 + 2KK_p)]} P_{a(t),b}(J) \exp[-t^2/4] dt,$$

which in turn can be rewritten (after multiplying and dividing by $Z_{a,b}$) as

$$P_{N, \text{Tr}^2}(J) = \frac{1}{\sqrt{4\pi}} \int_{\mathbb{R}} \frac{\mathbb{E}_{a,b} \exp\left[i\sqrt{\gamma}t(\text{Tr} \tilde{J} \tilde{J}^* - N(K_p + K))\right]}{\mathbb{E}_{a,b} \exp\left[-\gamma(\text{Tr} \tilde{J} \tilde{J}^* - N(K_p + K))^2\right]} P_{a(t),b}(J) \exp[-t^2/4] dt. \quad (25)$$

One advantage of the linearization is that the joint distribution of eigenvalues (also in the sense of a complex-valued measure) of $P_{a(t),b}$ can be given explicitly as

$$P_{a(t),b}(z) := \frac{1}{Z_{a(t),b}^{\text{EV}}} \prod_{j < l} |z_j - z_l|^2 \exp\left[-a(t) \sum_{j=1}^N |z_j|^2 + \frac{b}{2} \left(\sum_{j=1}^N z_j^2 + \overline{z_j^2}\right)\right], \quad (26)$$

where we abused notation by using the same symbol for the matrix and the eigenvalue distribution. The superscript EV in the normalization constant indicates that $Z_{a(t),b}$ and $Z_{a(t),b}^{\text{EV}}$ differ. The density (26) will be analyzed in the next section. By continuity it is clear that the joint distribution of eigenvalues of P_{N, Tr^2} also has a continuous density. Hence we can speak of its k -th correlation function ρ_{N, Tr^2}^k , which we define as in (4).

In terms of correlation functions, we get from (25)

$$\rho_{N, \text{Tr}^2}^k(z) = \frac{1}{\sqrt{4\pi}} \int_{\mathbb{R}} \frac{\mathbb{E}_{a,b} \exp \left[i\sqrt{\gamma}t(\text{Tr} \tilde{J} \tilde{J}^* - N(K_p + K)) \right]}{\mathbb{E}_{a,b} \exp \left[-\gamma(\text{Tr} \tilde{J} \tilde{J}^* - N(K_p + K))^2 \right]} \rho_{a(t),b}^k(z) \exp[-t^2/4] dt, \quad (27)$$

where $\rho_{a(t),b}^k$ denotes the k -th correlation function of $P_{a(t),b}$ and z is an abbreviation for z_1, \dots, z_k . Moreover,

$$\begin{aligned} & \rho_{N, \text{Tr}^2}^k(z) - \det(K_{\text{weak/strong}}(z_j, z_l))_{j,l \leq k} \\ &= \frac{1}{\sqrt{4\pi}} \int_{\mathbb{R}} \frac{\mathbb{E}_{a,b} e^{i\sqrt{\gamma}t(\text{Tr} \tilde{J} \tilde{J}^* - N(K_p + K))}}{\mathbb{E}_{a,b} e^{-\gamma(\text{Tr} \tilde{J} \tilde{J}^* - N(K_p + K))^2}} \left(\rho_{a(t),b}^k(z) - \det(K_{\text{weak/strong}}(z_j, z_l))_{j,l \leq k} \right) e^{-t^2/4} dt. \end{aligned} \quad (28)$$

Remark 6. From (27), we see the non-determinantality of the eigenvalue distribution of P_{N, Tr^2} . Its correlation functions are not determinants themselves, but rather averages of determinants.

3. ASYMPTOTICS FOR THE LINEARIZED ENSEMBLE

In this section, we will employ orthogonal polynomials and asymptotic analysis to obtain the asymptotic behavior of the linearized correlation functions $\rho_{a(t),b}^k$, as $N \rightarrow \infty$. For $\tau \in (0, 1)$, the orthogonal polynomials are Hermite polynomials, whereas for $\tau = 0$ the orthogonal polynomials are simple monomials. We will first concentrate on the much more involved case $\tau \neq 0$, the case $\tau = 0$ will be dealt with at the end of the proof of Proposition 9 below.

Let $(H_k)_{k \in \mathbb{N}}$ denote the sequence of Hermite polynomials, that is (cf. [1, 22.10.0])

$$H_k(z) := \frac{k!}{2\pi i} \oint \exp[-t^2 + 2zt] t^{-(k+1)} dt, \quad (29)$$

where the contour encircles the origin. It is well-known that these polynomials form an orthogonal sequence in $L^2(\mathbb{R})$ w.r.t. the weight $\exp(-t^2)$, i.e. for $k \neq l$

$$\int H_k(t) H_l(t) e^{-t^2} dt = 0.$$

Our analysis depends crucially on the fact that the Hermite polynomials are also orthogonal on the complex plane w.r.t. certain Gaussian measures, a fact first noticed in [27, 15]. This can be translated to complex weights easily. Recall first

$$a(t) = N \left(\frac{1}{1 - \tau^2} + 2\gamma K \right) + i\sqrt{\gamma}t, \quad b = \frac{\tau N}{1 - \tau^2},$$

and $a(0) = a$. Since b will remain unchanged, we will omit it in newly defined quantities to ease the notation.

Lemma 7. *Consider the function*

$$W_{a(t)}(z) := \exp \left[-a(t) |z|^2 + \frac{b}{2} (z^2 + \bar{z}^2) \right].$$

Then

$$\int_{\mathbb{C}} H_l(c_{a(t)}z) H_k(c_{a(t)}\bar{z}) W_{a(t)}(z) dz = \delta_{lk} \frac{k! \pi (2a(t))^k}{\sqrt{a(t)^2 - b^2} b^k}, \quad (30)$$

where

$$c_{a(t)} := \sqrt{\frac{a(t)^2 - b^2}{2b}}$$

and $\sqrt{\cdot}$ denotes the principal branch.

Proof. Using the integral representation (29) and the residue theorem one can easily verify (30) which we leave to the reader. \square

Define $(p_k)_{k \in \mathbb{N}}$ as the sequence of polynomials $p_k(z) = C_{a(t),k} H_k(c_{a(t)}z)$ with $C_{a(t),k} := \left(\frac{k! \pi (2a(t))^k}{\sqrt{a(t)^2 - b^2} b^k}\right)^{-1/2}$ such that

$$\int_{\mathbb{C}} p_l(z) p_k(\bar{z}) W_{a(t)}(z) dz = \delta_{lk}.$$

Now, using standard arguments, it is seen that the ensemble $P_{a(t),b}$ is determinantal, i.e. its correlation functions $\rho_{a(t),b}^k$, $k = 1, 2, \dots$,

$$\rho_{a(t),b}^k(z_1, \dots, z_k) = \frac{N!}{(N-k)!} \int_{\mathbb{C}^{N-k}} P_{a(t),b}(z) dz_{k+1} \dots dz_N$$

fulfill

$$\rho_{a(t),b}^k(z_1, \dots, z_k) = \det (K_{a(t)}(z_i, z_j))_{1 \leq i, j \leq k}$$

with the kernel

$$K_{a(t)}(z_1, z_2) := \sum_{j=0}^{N-1} p_j(z_1) p_j(\bar{z}_2) \sqrt{W_{a(t)}(z_1)} \sqrt{W_{a(t)}(\bar{z}_2)}. \quad (31)$$

The analysis of correlation functions of $P_{a(t),b}$ thus boils down to an analysis of the kernel $K_{a(t)}$. In the limit of weak non-Hermiticity we will prove

Proposition 8. Define $C_{\bar{K}} := 1 + 4\gamma\bar{K}$ with \bar{K} from Lemma 5. Let $X \in (-\frac{2}{\sqrt{C_{\bar{K}}}}, \frac{2}{\sqrt{C_{\bar{K}}}})$. As

$N \rightarrow \infty$, we have with $\tau = \tau_N := 1 - \frac{\tilde{\alpha}^2}{2C_{\bar{K}}^2 N}$, $\tilde{\alpha} > 0$,

$$\begin{aligned} \frac{1}{C_{\bar{K}}^2 N^2} K_{a(t)} \left(X + \frac{x_1 + iy_1}{C_{\bar{K}} N}, X + \frac{x_2 + iy_2}{C_{\bar{K}} N} \right) &= \frac{1}{\pi} \exp \left[-\frac{y_1^2 + y_2^2}{\tilde{\alpha}^2} + iX \frac{(y_1 - y_2)}{2} \right] \\ &\times \frac{1}{\sqrt{2\pi}\tilde{\alpha}} \int_{-\frac{1}{2}\sqrt{\frac{4}{C_{\bar{K}}} - X^2}}^{\frac{1}{2}\sqrt{\frac{4}{C_{\bar{K}}} - X^2}} \exp \left[-\frac{\tilde{\alpha}^2 u^2}{2} + iu(x_1 - x_2) - u(y_1 + y_2) \right] du + \mathcal{O} \left(\frac{\log N}{N} \right). \end{aligned}$$

The \mathcal{O} term is uniform for $X \in (-\frac{2}{\sqrt{C_{\bar{K}}}} + \delta, \frac{2}{\sqrt{C_{\bar{K}}}} - \delta)$ for any $\delta > 0$ fixed, any $x_j, y_j, j = 1, 2$ chosen from an arbitrary compact subset of \mathbb{R} and $|t| \leq 2\sqrt{\log N}$.

Remark. The reader will note that $\tilde{\alpha}$ in Proposition 8 differs from α in Theorem 3 by a factor $\nu(X)^{-1}$. The scaling of the local variables z_j differs also in the same way. This difference is purely due to notational convenience.

The limit of strong non-Hermiticity corresponds to a fixed $\tau \in [0, 1)$.

Proposition 9. *Let $\tau \in [0, 1)$ fixed and k be a nonnegative integer. As $N \rightarrow \infty$, we have with $C := \frac{1}{1-\tau^2} + 2\gamma K$*

$$\frac{1}{(CN)^k} \rho_{a(t), b}^k \left(Z + \frac{z_1}{\sqrt{CN}}, \dots, Z + \frac{z_k}{\sqrt{CN}} \right) = \det(K_{\text{strong}}(z_j, z_l)_{j, l \leq k} + \mathcal{O}\left(\frac{1}{\sqrt{N}}\right)),$$

where K_{strong} has been defined in (5). Here Z is chosen from the interior of the elliptic set

$$E := \left\{ Z \in \mathbb{C} : \frac{1-\tau+2\gamma K(1-\tau^2)}{1+\tau+2\gamma K(1-\tau^2)} \Re Z^2 + \frac{1+\tau+2\gamma K(1-\tau^2)}{1-\tau+2\gamma K(1-\tau^2)} \Im Z^2 \leq \frac{1}{C} \right\}.$$

The \mathcal{O} term is uniform for Z from any compact subset of E° , z_1, \dots, z_k from compacts of \mathbb{C} and $|t| \leq 2\sqrt{\log N}$.

A technical ingredient in the proof of Propositions 8 and 9 is stated in the following lemma for the normalized upper incomplete gamma function

$$Q(w, z) := \frac{\Gamma(w, z)}{\Gamma(w)}, \quad \Gamma(w, z) := \int_z^\infty t^{w-1} e^{-t} dt.$$

In these definitions w and z are real and positive but $\Gamma(w, z)$ and $Q(w, z)$ can in fact be continued to analytic functions in the complex plane, provided $w > 0$. We will use the same symbols for these continued functions.

Lemma 10 ([26]). *Denote $\eta := \sqrt{2(z-1-\log z)}$, where we choose the branch of the square root such that it has the same sign as $z-1$ for real positive z , and by continuity elsewhere. Then*

$$Q(N, Nz) = \frac{1}{2} \operatorname{erfc} \left(\eta \sqrt{(N/2)} \right) + \mathcal{O} \left(\frac{e^{-\frac{N}{2}\eta^2}}{\sqrt{N}} \right), \quad (32)$$

where erfc denotes the complementary error function

$$\operatorname{erfc}(z) := \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2} dt,$$

and the \mathcal{O} term is uniform in the domain $|\arg(z)| \leq 2\pi - \delta$ with arbitrary $\delta > 0$, i.e. for $z = re^{i\varphi}$ with $-2\pi + \delta \leq \varphi \leq 2\pi - \delta$, $r > 0$.

Proof. The lemma is a special case of an asymptotic expansion derived in [26], where it is shown that

$$Q(N, Nz) = \frac{1}{2} \operatorname{erfc} \left(\eta \sqrt{(N/2)} \right) + R_N(\eta)$$

and the remainder $R_N(\eta)$ in (32) admits an asymptotic expansion in negative powers of N as $N \rightarrow \infty$,

$$R_N(\eta) \sim (2\pi N)^{-1/2} e^{-\frac{1}{2}N\eta^2} \sum_{k=0}^{\infty} c_k(\eta) N^{-k}.$$

The expansion is uniform for η in the domain $|\arg(z)| \leq 2\pi - \delta$ with δ being an arbitrarily small positive constant. \square

Before beginning with the proof of Proposition 8, we state the relevant asymptotics of the complementary error function for later use. From [1, 6.5.32] we have

$$\operatorname{erfc}(w) = \frac{e^{-w^2}}{\sqrt{\pi}w} (1 + \mathcal{O}(w^{-2})), \quad (33)$$

as $w \rightarrow \infty$ in $|\arg(w)| < 3\pi/4$. Moreover, we have by the relation $\operatorname{erfc}(-w) = 2 - \operatorname{erfc}(w)$

$$\operatorname{erfc}(-w) = 2 - \frac{e^{-w^2}}{\sqrt{\pi}w} (1 + \mathcal{O}(w^{-2})), \quad w \rightarrow \infty, \quad (34)$$

again with $|\arg(w)| < 3\pi/4$.

Proof of Proposition 8. We start with an alternative representation of the Hermite polynomials (cf. [1, 22.10.15] for the sign $(-2i)^k$, for the other sign use the parity $H_k(-z) = (-1)^k H_k(z)$),

$$H_k(z) = \frac{(\pm 2i)^k}{\sqrt{\pi}} \exp[z^2] \int_{\mathbb{R}} r^k \exp[-r^2 \mp 2izr] dr,$$

which allows to rewrite the kernel as

$$\begin{aligned} K_{a(t)}(z_1, z_2) &= \frac{\sqrt{a(t)^2 - b^2}}{\pi^2} \exp \left[c_{a(t)}^2 (z_1^2 + \overline{z_2}^2) - \frac{a(t)}{2} (|z_1|^2 + |z_2|^2) + \frac{b}{4} (z_1^2 + z_2^2 + \overline{z_1}^2 + \overline{z_2}^2) \right] \\ &\times \sum_{j=0}^{N-1} \left(\frac{2b}{a(t)} \right)^j \frac{1}{j!} \int_{\mathbb{R}} \int_{\mathbb{R}} (rs)^j \exp[-(r^2 + s^2) + 2ic_{a(t)}(rz_1 - s\overline{z_2})] dr ds. \end{aligned} \quad (35)$$

Choosing

$$z_j = X + \frac{x_j}{N} + i \frac{y_j}{N}, \quad j = 1, 2$$

with $X, x_j, y_j \in \mathbb{R}$, and decoupling the integrals via the substitution

$$\Phi(r, s) := (u, v) := \left(\frac{r+s}{c_{a(t)}}, \frac{r-s}{c_{a(t)}} \right), \quad |\det D\Phi(r, s)| = \frac{2}{|c_{a(t)}|^2},$$

we arrive at

$$\begin{aligned}
& K_{a(t)}(z_1, z_2) \\
&= \frac{\sqrt{a(t)^2 - b^2} |c_{a(t)}|^2}{2\pi^2} \exp \left[\left(c_{a(t)}^2 - \frac{a(t) - b}{2} \right) \frac{x_1^2 + x_2^2}{N^2} - \left(c_{a(t)}^2 + \frac{a(t) + b}{2} \right) \frac{y_1^2 + y_2^2}{N^2} \right. \\
&\quad \left. + \frac{2ic_{a(t)}^2 X(y_1 - y_2)}{N} + \left(2c_{a(t)}^2 - a(t) + b \right) \frac{X(x_1 + x_2)}{N} + \frac{2ic_{a(t)}^2 (x_1 y_1 - x_2 y_2)}{N^2} \right] \\
&\times \int_{\mathbb{R}/c_{a(t)}} \exp \left[-\frac{c_{a(t)}^2 u^2}{2} \left(1 - \frac{b}{a(t)} \right) + \frac{ic_{a(t)}^2 u(x_1 - x_2)}{N} - \frac{c_{a(t)}^2 u(y_1 + y_2)}{N} \right] \\
&\times \int_{\mathbb{R}/c_{a(t)}} \exp \left[-\frac{c_{a(t)}^2 v^2}{2} \left(1 + \frac{b}{a(t)} \right) + 2ic_{a(t)}^2 vX + (2c_{a(t)}^2 - a(t) + b)X^2 \right. \\
&\quad \left. - \frac{c_{a(t)}^2 v(y_1 - y_2)}{N} + \frac{ic_{a(t)}^2 v(x_1 + x_2)}{N} \right] Q \left(N, \frac{c_{a(t)}^2 b}{2a(t)} (u^2 - v^2) \right) dv du. \tag{36}
\end{aligned}$$

Here, Q is the regularized upper incomplete gamma function. To see that (36) holds, observe that for w being a positive integer

$$Q(w, z) = e^{-z} \sum_{j=0}^{w-1} \frac{z^j}{j!}, \tag{37}$$

an identity which can be easily derived from the recurrence relations of $\Gamma(w, z)$.

Let us consider the v -integral in (36). Using the definition of $c_{a(t)}$ in Lemma 7, it can be written as

$$\begin{aligned}
& \int_{\mathbb{R}/c_{a(t)}} \exp \left[-\frac{c_{a(t)}^2 v^2}{2} \left(1 + \frac{b}{a(t)} \right) + 2ic_{a(t)}^2 vX + (2c_{a(t)}^2 - a(t) + b)X^2 \right. \\
&\quad \left. - \frac{c_{a(t)}^2 v(y_1 - y_2)}{N} + \frac{ic_{a(t)}^2 v(x_1 + x_2)}{N} \right] Q \left(N, \frac{c_{a(t)}^2 b}{2a(t)} (u^2 - v^2) \right) dv \\
&= \int_{\mathbb{R}/c_{a(t)}} \exp \left[-\frac{(a(t) + b)^2 (a(t) - b)}{4a(t)b} \left(v - \frac{2a(t)}{a(t) + b} iX \right)^2 - \frac{c_{a(t)}^2 v(y_1 - y_2)}{N} \right. \\
&\quad \left. + \frac{ic_{a(t)}^2 v(x_1 + x_2)}{N} \right] Q \left(N, \frac{c_{a(t)}^2 b}{2a(t)} (u^2 - v^2) \right) dv. \tag{38}
\end{aligned}$$

We see that the integrand has a saddle point at $v = 2a(t)iX/(a(t) + b) = iX + \mathcal{O}(1/N)$. We note in passing that it is important to perform the saddle point approximation of the integral around the exact location of the saddle point and not at its asymptotic location $v = iX$, as in the latter it does not become clear whether any subleading v^2 or v terms have been missed that may become leading after applying the saddle point approximation.

In applying the saddle point method, some care is needed as $Q(N, c_{a(t)}^2 b(u^2 - v^2)/(2a(t)))$ depends on N . To see that the main contribution to the integral (38) comes from a neighborhood of $v = 2a(t)iX/(a(t) + b)$, we first note the asymptotics

$$\frac{(a(t) + b)^2(a(t) - b)}{4a(t)b} = \frac{NC_{\bar{K}}}{2} + \mathcal{O}(1), \quad (39)$$

$$\frac{c_{a(t)}^2 b}{2a(t)} = \frac{NC_{\bar{K}}}{4} + \mathcal{O}(1), \quad (40)$$

where we recall $C_{\bar{K}} = 1 + 4\gamma\bar{K}$. Note that both $\mathcal{O}(1)$ terms are in general complex-valued.

We will first show that the v -integral can be truncated to a compact interval. Computing $c_{a(t)}^2 = NC_{\bar{K}}/2 + \mathcal{O}(1)$, we see that the terms

$$-\frac{c_{a(t)}^2 v(y_1 - y_2)}{N} + \frac{ic_{a(t)}^2 v(x_1 + x_2)}{N}$$

in (32) are of order 1 and linear in v and hence play no role in the truncation argument. Using Lemma 10 we find that, depending on the location of η in the complex plane, one or both of the formulae (33) and (34) are applicable. In any case, we need to control the term $\exp(-\eta^2 N/2) = \exp(-N(z - 1 - \log z))$ with

$$z = \frac{C_{\bar{K}}}{4}(u^2 - v^2)(1 + \mathcal{O}(1/N)).$$

To do that, note that for $\Re z > -c$ for some small $c > 0$ we have $\Re(z - 1 - \log z) > 0$ or equal $+\infty$ at 0 and hence for such z we get $|e^{-\eta^2 N/2}| \leq 1$. For z with $\Re z \leq -c$ we have

$$\Re(z - 1 - \log z) \geq (1 + \varepsilon)\Re z - C_\varepsilon \geq -(1 + \varepsilon)\frac{C_{\bar{K}}}{4}(\Re v)^2 - C_\varepsilon$$

for $\varepsilon > 0$ small and some $C_\varepsilon > 0$. Here we used that the real part of a point on $\mathbb{R}/c_{a(t)}$ is (in absolute values) larger than its imaginary part. This corresponds to the fact that as $N \rightarrow \infty$, the slope of the line $\mathbb{R}/c_{a(t)}$ tends to 0. Comparing with (38) and (39), we see that the growth of $\exp(-\eta^2 N/2)$ is suppressed by the fast decay of $\exp(-c_{a(t)}^2 v^2(1 + b/a(t))/2)$. This shows on the one hand that for N large enough, we can by analyticity change the integration line $\mathbb{R}/c_{a(t)}$ to \mathbb{R} , which can on the other hand be truncated to $[-R, R]$ for some $R > 0$ large enough, the remaining part of the integral being exponentially small. Our estimates show that R can be chosen to be independent of u .

Again by analyticity, we can deform the path of integration as follows. We first integrate from $-R$ to $-R + i\frac{2a(t)}{a(t) + b}X$, from there to $R + i\frac{2a(t)}{a(t) + b}X$ and then to R . The same argument as above can be used to see that the integrals along the segments paralleling the imaginary axis can be neglected, yielding only an exponentially small term that is independent of u .

The remaining integral can be written as

$$\begin{aligned}
& \int_{-R}^R \exp \left[-\frac{(a(t)+b)^2(a(t)-b)}{4a(t)b} v^2 - \frac{c_{a(t)}^2(v + i\frac{2a(t)}{a(t)+b}X)(y_1 - y_2)}{N} \right. \\
& \quad \left. + \frac{ic_{a(t)}^2(v + i\frac{2a(t)}{a(t)+b}X)(x_1 + x_2)}{N} \right] Q \left(N, \frac{c_{a(t)}^2 b}{2a(t)} \left(u^2 - (v + i\frac{2a(t)}{a(t)+b}X)^2 \right) \right) dv \\
&= \exp \left[-C_{\bar{K}} X \left(i\frac{y_1 - y_2}{2} + \frac{x_1 + x_2}{2} \right) \right] \int_{-R}^R \exp \left[-\frac{NC_{\bar{K}}}{2} v^2 - C_{\bar{K}} v \frac{y_1 - y_2}{2} + iC_{\bar{K}} v \frac{(x_1 + x_2)}{2} \right] \\
&\times \exp \left[\mathcal{O} \left(\frac{|v| + |x_1| + |x_2| + |y_1| + |y_2|}{N} + v^2 \right) \right] \\
&\times Q \left(N, \frac{NC_{\bar{K}}}{4} (u^2 + X^2 - v^2 - 2ivX)(1 + \mathcal{O}(1/N)) \right) dv. \tag{41}
\end{aligned}$$

Since $x_i, y_i = \mathcal{O}(1), i = 1, 2$, we have

$$\exp [\mathcal{O}((|x_1| + |x_2| + |y_1| + |y_2|)/N)] = 1 + \mathcal{O}(1/N).$$

A closer look at the above truncation argument reveals that v is concentrated in an interval $(-\mathcal{O}(1/\sqrt{N}), \mathcal{O}(1/\sqrt{N}))$. Hence assuming smallness of $|v|$, say $|v| \leq N^{-1/2-\delta}$ for some $\delta > 0$, results in an exponentially small error. We will apply Lemma 10 together with (33) and (34) to

$$z = \frac{C_{\bar{K}}}{4} (u^2 + X^2 - v^2 - 2ivX)(1 + \mathcal{O}(1/N))$$

in order to show that

$$Q \left(N, \frac{NC_{\bar{K}}}{4} (u^2 + X^2 - v^2 - 2ivX)(1 + \mathcal{O}(1/N)) \right) = \mathbb{1}_{\{u^2 + X^2 \leq 4/C_{\bar{K}}\}} + \mathcal{O} \left(\frac{\log N}{N} \right) \tag{42}$$

holds uniformly in v and u . Let us first consider those z well bounded away from 1. For these z , we have $|\eta\sqrt{N/2}| \rightarrow \infty$ and hence (33) or (34) are applicable. If $\Re(z - 1) > \log N/N$, then (33) gives $Q(N, Nz) = \mathcal{O}(1/N)$. If $-1 - c < \Re(z - 1) < -\log N/N$ with $c > 0$ small, then by the choice of the square root, $\Re\eta < 0$ and (34) yields $Q(N, Nz) = 1 + \mathcal{O}(1/N)$.

For the region $|\Re z - 1| \leq \log N/N$, we argue that by analyticity of η and erfc and by (33) and (34) for those z with $N|\eta|^2 \rightarrow \infty$, we get that $Q(N, Nz)$ is bounded uniformly in N . Hence this region gives a contribution of order $\mathcal{O}(\log N/N)$.

It is now straightforward to deduce that (41) equals

$$\sqrt{\frac{2\pi}{NC_{\bar{K}}}} \exp \left[-iC_{\bar{K}} X \frac{y_1 - y_2}{2} - C_{\bar{K}} X \frac{x_1 + x_2}{2} \right] \mathbb{1}_{\{u^2 + X^2 \leq 4/C_{\bar{K}}\}} \left(1 + \mathcal{O} \left(\frac{\log N}{N} \right) \right) \tag{43}$$

uniformly in u . The statement of the proposition now follows from combining (43) with (36) and the asymptotics

$$\sqrt{a(t)^2 - b^2} |c_{a(t)}|^2 = \frac{N^{5/2} C_{\bar{K}}^{5/2} + \mathcal{O}(N^{3/2})}{2\tilde{\alpha}}, \quad (44)$$

$$\begin{aligned} c_{a(t)}^2 - \frac{a(t)}{2} + \frac{b}{2} &= \frac{a(t)(a(t) - b)}{2b} = \frac{N}{4} C_{\bar{K}} + \mathcal{O}(1), \\ c_{a(t)}^2 + \frac{a(t)}{2} + \frac{b}{2} &= \frac{(a(t) + b)^2}{2b} = \frac{N^2 C_{\bar{K}}^2}{\tilde{\alpha}^2} + \mathcal{O}(N), \\ c_{a(t)}^2 &= \frac{N}{2} C_{\bar{K}} + \mathcal{O}(1), \end{aligned} \quad (45)$$

$$1 - \frac{b}{a(t)} = \frac{\tilde{\alpha}^2}{2C_{\bar{K}}N} + \mathcal{O}\left(\frac{1}{N^2}\right). \quad (46)$$

□

Proof of Proposition 9. For the limit of strong non-Hermiticity, we choose

$$z_j = X + iY + \frac{x_j}{\sqrt{N}} + i\frac{y_j}{\sqrt{N}}, \quad j = 1, 2$$

with $X, Y, x_j, y_j \in \mathbb{R}$. We deal with the more complicated case $\tau \neq 0$ first. Analogously to (36), we obtain

$$\begin{aligned} K_{a(t)}(z_1, z_2) &= \frac{\sqrt{a(t)^2 - b^2} c_{a(t)}^2}{2\pi^2} \exp \left[-\frac{a(t)(x_1^2 + x_2^2)}{2N} - \frac{a(t)(y_1^2 + y_2^2)}{2N} + \frac{i(a(t) - b)X(y_1 - y_2)}{\sqrt{N}} \right. \\ &\quad \left. - \frac{i(a(t) + b)Y(x_1 - x_2)}{\sqrt{N}} + (2ic_{a(t)}^2 - i\frac{a(t)^2}{b}) \frac{(x_1 y_1 - x_2 y_2)}{N} + \frac{a(t)(x_1 x_2 + y_1 y_2 - ix_1 y_2 + ix_2 y_1)}{N} \right] \\ &\times \int_{\mathbb{R}/c_{a(t)}} \exp \left[-\frac{(a(t) - b)^2(a(t) + b)}{4a(t)b} \left(u + \frac{a(t)}{a(t) - b} \left(2Y - \frac{i(x_1 - x_2)}{\sqrt{N}} + \frac{y_1 + y_2}{\sqrt{N}} \right) \right)^2 \right] \\ &\times \int_{\mathbb{R}/c_{a(t)}} \exp \left[-\frac{(a(t) + b)^2(a(t) - b)}{4a(t)b} \left(v - \frac{a(t)}{a(t) + b} \left(2iX + \frac{i(x_1 + x_2)}{\sqrt{N}} - \frac{y_1 - y_2}{\sqrt{N}} \right) \right)^2 \right] \\ &\times Q \left(N, \frac{c_{a(t)}^2 b}{2a(t)} (u^2 - v^2) \right) dv du. \end{aligned} \quad (47)$$

We can now proceed analogously to the proof of Proposition 8. For the v -integral we get

$$\begin{aligned} &\int_{\mathbb{R}/c_{a(t)}} \exp \left[-\frac{(a(t) + b)^2(a(t) - b)}{4a(t)b} \left(v - \frac{a(t)}{a(t) + b} \left(2iX + \frac{i(x_1 + x_2)}{\sqrt{N}} - \frac{y_1 - y_2}{\sqrt{N}} \right) \right)^2 \right] \\ &\times \frac{1}{\sqrt{2\pi}} \sqrt{\frac{(a(t) + b)^2(a(t) - b)}{2a(t)b}} Q \left(N, \frac{c_{a(t)}^2 b}{2a(t)} (u^2 - v^2) \right) dv \\ &= \mathbb{1}_{\frac{c_{a(t)}^2 b}{2aN} u^2 + \frac{a(a-b)}{(a+b)N} X^2 \leq 1} \left(1 + \mathcal{O}(1/\sqrt{N}) \right), \end{aligned}$$

where $a = a(0)$. Recall from the proof of Proposition 8 that the error bounds can be chosen uniform in u . Thus the same procedure can be repeated for the u -integral, giving

$$\begin{aligned} & \int_{\mathbb{R}/c_{a(t)}} \exp \left[- \frac{(a(t) - b)^2(a(t) + b)}{4a(t)b} \left(u + \frac{a(t)}{a(t) - b} \left(2Y - \frac{i(x_1 - x_2)}{\sqrt{N}} + \frac{y_1 + y_2}{\sqrt{N}} \right) \right)^2 \right] \\ & \times \frac{1}{\sqrt{2\pi}} \sqrt{\frac{(a(t) - b)^2(a(t) + b)}{2a(t)b}} \mathbb{1}_{\frac{c_{a(t)}^2}{2a(t)}u^2 + \frac{a(a-b)}{(a+b)N}X^2 \leq 1} \left(1 + \mathcal{O}(1/\sqrt{N}) \right) du \\ & = \mathbb{1}_{\frac{a(a-b)}{(a+b)N}X^2 + \frac{a(a+b)}{(a-b)N}Y^2 \leq 1} + \mathcal{O}(1/\sqrt{N}). \end{aligned} \quad (48)$$

To obtain the final form of the proposition, note that the determinant is invariant of conjugations of the kernel, i.e. for any kernel K

$$\det(K(z_j, z_l)) = \det(\tilde{K}(z_j, z_l)),$$

where $\tilde{K}(z_j, z_l) := K(z_j, z_l)f(z_j)/f(\bar{z}_l)$ and f is some function without zeros or singularities. Using this, we see that the exponential factors

$$\exp \left[\frac{i(a(t) - b)X(y_1 - y_2)}{\sqrt{N}} - \frac{i(a(t) + b)Y(x_1 - x_2)}{\sqrt{N}} + (2ic_{a(t)}^2 - i\frac{a(t)^2}{b})\frac{(x_1y_1 - x_2y_2)}{N} \right]$$

cancel when taking the determinant.

Let us now consider the case $\tau = 0$. Then $b = 0$ and it can be easily checked using polar coordinates that the orthonormal polynomials to the weight function $W_{a(t)}$ of Lemma 7 are

$$p_j(z) := \sqrt{\frac{a(t)^{j+1}}{\pi j!}} z^j,$$

$\sqrt{\cdot}$ denoting the principal branch. This gives with (31)

$$K_{a(t)}(z_1, z_2) = \frac{a(t)}{\pi} \exp \left[-\frac{a(t)}{2}(|z_1|^2 + |z_2|^2 - 2z_1\bar{z}_2) \right] Q(N, a(t)z_1\bar{z}_2).$$

Invoking the asymptotics (42), it is straightforward to finish the proof of the proposition. \square

4. PROOF OF MAIN THEOREMS

We will prove both main results simultaneously.

Proof of Theorem 1 and Theorem 3. We will start with proving parts b) of both theorems. Recall from (28)

$$\begin{aligned} & \rho_{N, \text{Tr}^2}^k(z) - \det(K_{\text{weak, strong}}(z_j, z_l))_{j, l \leq k} \\ & = \frac{1}{\sqrt{4\pi}} \int_{\mathbb{R}} \frac{\mathbb{E}_{a,b} e^{i\sqrt{\gamma}t(\text{Tr} \tilde{J} \tilde{J}^* - N(K_p + K))}}{\mathbb{E}_{a,b} e^{-\gamma(\text{Tr} \tilde{J} \tilde{J}^* - N(K_p + K))^2}} \left(\rho_{a(t), b}^k(z) - \det(K_{\text{weak, strong}}(z_j, z_l))_{j, l \leq k} \right) e^{-t^2/4} dt, \end{aligned} \quad (49)$$

where $K_{\text{weak, strong}}$ means either K_{weak} or K_{strong} . By Proposition 8 with $C := C_{\tilde{K}}$ and $\tilde{\alpha} := C\alpha/\nu(X)$ or Proposition 9 we have convergence of the term in the parenthesis to 0 with the error prescribed in the theorems, uniform for $|t| \leq 2\sqrt{\log N}$. Note here that the phase factor of the limiting kernel of Proposition 8 cancels when taking determinants. By (10) and

Lemma 5 we have that $\mathbb{E}_{a,b} e^{-\gamma(\text{Tr} \tilde{J} \tilde{J}^* - N(K_p + K))^2}$ is bounded away from 0 uniformly in N . Clearly, $\left| \mathbb{E}_{a,b} e^{i\sqrt{\gamma}t(\text{Tr} \tilde{J} \tilde{J}^* - N(K_p + K))} \right|$ is bounded above by 1. For $|t| > 2\sqrt{\log N}$, we use

$$\begin{aligned} & \mathbb{E}_{a,b} \exp \left[i\sqrt{\gamma}t(\text{Tr} \tilde{J} \tilde{J}^* - N(K_p + K)) \right] P_{a(t),b}(J) \\ &= \exp \left[-i\sqrt{\gamma}tN(K_p + K) \right] P_{a,b}(J) \exp \left[i\sqrt{\gamma}t \text{Tr} J J^* \right] \end{aligned}$$

and consequently

$$\left| \mathbb{E}_{a,b} e^{i\sqrt{\gamma}t(\text{Tr} \tilde{J} \tilde{J}^* - N(K_p + K))} \rho_{a(t),b}^k \right| \leq \rho_{a,b}^k. \quad (50)$$

The uniformity of the convergence to the bounded limiting kernel in Proposition 8 or Proposition 9 thus proves the boundedness of the integrand of (49) in t and N . Hence we can split up the t -integral into $|t| \leq 2\sqrt{\log N}$ and $|t| > 2\sqrt{\log N}$. The first integral gives the desired result whereas the second integral is $\mathcal{O}(1/N)$ by (33).

For the remaining part of the proof, we concentrate on the more complicated case $\tau \neq 0$. Part a) of Theorem 1 follows for $Z \in E^\circ$ immediately from b). For $Z \notin E$, the statement follows from (48) together with (27) and (50).

To prove part a) of Theorem 3, note that (7) follows (formally) directly from b): Choosing $\alpha = \sqrt{2\kappa}\nu(X)$, $k = 1$ and $z_1 = z_2 = iy$, we obtain

$$\begin{aligned} & \int \frac{1}{N} \rho_{N,\text{Tr}^2}^1(X + iY) dY = \int \frac{1}{N^2 \nu(X)} \rho_{N,\text{Tr}^2}^1 \left(X + \frac{iy}{N \nu(X)} \right) dy \\ & \rightarrow \nu(X) \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp \left[-\frac{\alpha^2 u^2}{2} \right] \int_{\mathbb{R}} \frac{\sqrt{2}}{\sqrt{\pi} \alpha} \exp \left[-\frac{2y^2}{\alpha^2} - 2uy \right] dy du = \nu(X), \quad N \rightarrow \infty. \end{aligned}$$

To make this argument rigorous, we need to show interchangeability of limit and integration. In view of (27), Lemma 5 and (50), it suffices to show uniform integrability of $y \mapsto N^{-2} \rho_{a,b}^1(X + iy/N)$. With (36), (41) and (43), we get with $z := X + iy/N$

$$\begin{aligned} K_a(z, z) &= \frac{\sqrt{a^2 - b^2} |c_a|^2}{2\pi^2} \exp \left[-2 \left(c_a^2 + \frac{a+b}{2} \right) \frac{y^2}{N^2} \right] \\ &\times \int_{\mathbb{R}} \exp \left[-\frac{c_a^2 u^2}{2} \left(1 - \frac{b}{a} \right) - \frac{2c_a^2 u y}{N} \right] \sqrt{\frac{2\pi}{NC_{\tilde{K}}}} \mathbb{1}_{\{u^2 + X^2 \leq 4/C_{\tilde{K}}\}} \left(1 + \mathcal{O} \left(\frac{\log N}{N} \right) \right) du, \end{aligned}$$

where the \mathcal{O} -term does not depend on y (as the bound is uniform in u and the v -integral does not depend on y as here $y_1 = y_2$). Because of the indicator function, we can assume $|u| \leq M$ in the integral for some M . By the asymptotics (44), (45) and (46), it follows that for some constants $\tilde{C}_j > 0, j = 1, \dots, 4$ not depending on N , we have the crude bound

$$\begin{aligned} \left| \frac{1}{N^2} K_a(z, z) \right| &\leq \tilde{C}_1 \exp \left[-\tilde{C}_2 y^2 + \tilde{C}_3 M y \right] \int_{-M}^M \exp \left[-\frac{c_a^2 u^2}{2} \left(1 - \frac{b}{a} \right) \right] du \\ &\leq \tilde{C}_4 \exp \left[-\tilde{C}_2 y^2 + \tilde{C}_3 M y \right]. \end{aligned}$$

The last bound is clearly integrable in y which completes the proof of (7).

Finally, to prove (6), it suffices, analogously to above, to consider $K_a(Z, Z)$, $Z = X + iY$, $Y \neq 0$. From (47), we get

$$K_a(Z, Z) = \frac{\sqrt{a^2 - b^2} c_a^2}{2\pi^2} \int_{\mathbb{R}} \exp \left[- \frac{(a-b)^2(a+b)}{4ab} \left(u + \frac{2a}{a-b} Y \right)^2 \right] \\ \times \int_{\mathbb{R}} \exp \left[- \frac{(a+b)^2(a-b)}{4ab} \left(v - \frac{2ia}{a+b} X \right)^2 \right] Q \left(N, \frac{c_{a(t)}^2 b}{2a(t)} (u^2 - v^2) \right) dv du.$$

In contrast to the strongly non-Hermitian situation, here the term $2a/(a-b)$ in front of Y is of order N which leads after a shift to an expression $Q(N, C_N)$ with $C_N > 0$ of order N^2 . Now, (32) and (33) give the result. \square

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